JOHN P. GAPCYNSKI

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ON A FUNCTION-THEORY METHOD FOR OBTAINING POTENTIAL-FLOW PATTERNS OF A COMPRESSIBLE FLUID

By Abe Gelbart

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

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ADVANCE RESTRICTED REPORT

ON A FUNCTION-THEORY METHOD FOR OBTAINING POTENTIAL-FLOW PATTERNS OF A COMPRESSIBLE FLUID

By Abe Gelbart

SUMMARY

A scheme for obtaining exact potential-flow patterns in a compressible fluid is presented. The method is based on a complex-function theory developed recently for the solutions of the simultaneous first-order partial differential equations in the hodograph variables. The procedure suggested is to take a given incompressible-flow pattern given by an analytic function and to replace this function by an associated complex function, a solution of the compressible-flow equations, which will represent an associated compressible-flow pattern. This method formally solves the problem for obtaining an exact flow past a body in a compressible fluid; however, before such general flow patterns can be obtained, the new complex functions involved must first be studied and tabulated.

INTRODUCTION

This paper is intended to outline or sketch a process for creating flow patterns of a compressible fluid by means of a generalized concept of a complex variable. It is known that the present modes of treating this problem are essentially of an approximate nature. For example, the methods of Prandtl and Glauert, Ackeret, Poggi, Janzen and Rayleigh, and others are of an iterative nature and, after one or two steps, become unmanageable. Recently Ringleb, following Chaplygin's original memoir published in 1904, obtained exact solutions of the differential equations for compressible flows corresponding to a source and a vortex. Ringleb's approach, however, does not appear to yield a general process for handling the problem.

It is believed that the method outlined in this paper is a natural approach to the solution of the problem. The mathematical background for the details of the method has already been developed. (See reference 1.) Only those steps essential to the process are given herein.

THEORETICAL BACKGROUND

It is well known that the relations between the potential function ϕ and the stream function ψ in the incompressible case are

$$\left.\begin{aligned} \phi_x &= \psi_y \\ \phi_y &= -\psi_x \end{aligned}\right\} \quad (P_i) \qquad (1)$$

These equations will be referred to as the P_i (physical, incompressible) equations. Since equations (1) are the Cauchy-Riemann equations, $f = \phi + i\psi$ is an analytic function of a complex variable $z = (x + iy)$. It follows then that the reflected velocity vector is

$$\frac{df}{dz} = u - iv = qe^{-i\theta} \qquad (2)$$

where q is the magnitude of the velocity and θ is the angle the velocity vector makes with the x-axis.

If q and θ are introduced as independent variables, equations (1) take on the form

$$\left.\begin{aligned} \phi_\theta &= q\psi_q \\ \phi_q &= -\frac{1}{q}\psi_\theta \end{aligned}\right\} \quad (H_i) \qquad (3)$$

These equations will be referred to as the H_i (hodograph, incompressible) equations.

The equations corresponding to equations (1), in the compressible case, are

$$\left.\begin{aligned} \phi_x &= \frac{\rho_o}{\rho}\psi_y \\ \phi_y &= -\frac{\rho_o}{\rho}\psi_x \end{aligned}\right\}(P_c) \qquad (4)$$

where ρ is the density of the fluid and ρ_o the stagnation $(q = 0)$ density. These equations will be referred to as the P_c (physical, compressible) equations. Since ρ is a function of ϕ_x and ϕ_y, equations (4) are nonlinear in character and therefore are, in general, too difficult to handle. It was first noticed by Molenbroeck and later by Chaplygin that, if θ and q are chosen as the independent variables, then the equations corresponding to equations (4) are linear in character. In the new independent variables the equations become

$$\left.\begin{aligned} \phi_\theta &= \frac{\rho_o q}{\rho}\psi_q \\ \phi_q &= -\frac{\rho_o(1-M^2)}{\rho q}\psi_\theta \end{aligned}\right\}(H_c) \qquad (5)$$

where $M^2 = q^2/a^2$ and a is the velocity of sound corresponding to q.

It is remarked that for given stagnation conditions both ρ and M are functions of q only. Thus, the coefficients of ψ_q and ψ_θ are functions of q only, and equations (5) are therefore linear. The equations for ρ and M^2 are

$$\left.\begin{aligned} \rho &= \rho_o\left(1-\frac{\gamma-1}{2}\frac{q^2}{a_o^2}\right)^{\frac{1}{\gamma-1}} \\ M^2 &= \frac{q^2}{1-\frac{\gamma-1}{2}\frac{q^2}{a_o^2}} \end{aligned}\right\} \qquad (6)$$

where

a_o stagnation velocity of sound

γ ratio of specific heats at constant volume and constant pressure

It has been proved in reference 1 that, if $\phi(\theta,q)$ and $\psi(\theta,q)$ satisfy equations (5), then

$$F(\theta,q) = \Phi + i\Psi = \int_{\theta_1,q_1}^{\theta,q} \left(\phi\, d\theta - \frac{\rho_o(1-M^2)}{\rho q}\,\psi\, dq\right)$$

$$+ i\int_{\theta_1,q_1}^{\theta,q} \left(\psi\, d\theta + \frac{\rho}{\rho_o q}\,\phi\, dq\right) \qquad (7)$$

is a complex function of the end point (θ,q), independent of the path of integration, whose real and imaginary parts Φ and Ψ also satisfy equations (5). The lower limits θ_1 and q_1 may be arbitrarily chosen. For most purposes it is convenient to choose $q_1 = q_m$, where q_m is the maximum possible velocity, given by

$$q_m = \sqrt{\frac{2}{\gamma - 1}}\, a_o \approx \sqrt{5}\, a_o \qquad (8)$$

If, now, $f = \phi + i\psi = 1 + i0$, then the line integral of equation (7) yields

$$\Phi + i\Psi = W^{(1)} = \theta + i\int_{q_1}^{q} \frac{\rho}{\rho_o q}\, dq \qquad (9)$$

By repeated application of the foregoing process an unlimited number of particular solutions of equations (5) can be obtained. Thus

$$\left.\begin{aligned}
W^{(2)} &= \theta^2 - 2!\int_{q_1}^{q} \frac{\rho_o(1-M^2)}{\rho q}\int_{q_1}^{q} \frac{\rho}{\rho_o q}\,dq^2 + 2i\theta\int_{q_1}^{q}\frac{\rho}{\rho_o q}\,dq \\
W^{(3)} &= \theta^3 + 3i\theta^2\int_{q_1}^{q}\frac{\rho}{\rho_o q}\,dq - 3\theta 2!\int_{q_1}^{q}\frac{\rho_o(1-M^2)}{\rho q}\int_{q_1}^{q}\frac{\rho}{\rho_o q}\,dq^2 \\
&\quad - i3!\int_{q_1}^{q}\frac{\rho}{\rho_o q}\int_{q_1}^{q}\frac{\rho_o(1-M^2)}{\rho q}\int_{q_1}^{q}\frac{\rho}{\rho_o q}\,dq^3 \\
&\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
W^{(n)} &= \ldots
\end{aligned}\right\} \quad (10)$$

(See reference 1.)

Similarly, when $f = 0 + i$, a complementary set of particular solutions of equations (5) are obtained. These take on the form

$$\left.\begin{aligned}
i\tilde{W}^{(1)} &= i\left(\theta + i\int_{q_1}^{q}\frac{\rho_o(1-M^2)}{\rho q}\,dq\right) \\
i\tilde{W}^{(2)} &= i\left(\theta^2 - 2!\int_{q_1}^{q}\frac{\rho}{\rho_o q}\int_{q_1}^{q}\frac{\rho_o(1-M^2)}{\rho q}\,dq^2\right. \\
&\qquad \left. + 2i\theta\int_{q_1}^{q}\frac{\rho_o(1-M^2)}{\rho q}\,dq\right) \\
&\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
i\tilde{W}^{(n)} &= \ldots
\end{aligned}\right\} \quad (11)$$

(See reference 1.) If γ is chosen as 7/5, the evaluation of the integrals is simplified.

When the fluid is incompressible (that is, $a = \infty$, or $M = 0$, $\rho = \rho_o$), the solutions $W^{(n)}$ and $i\tilde{W}^{(n)}$ reduce to solutions of equations (3) and become, respectively, $(\theta + i \log q)^n$ and $i(\theta + i \log q)^n$. That these are solutions of equations (3) can easily be verified. It is precisely this correspondence between the solutions of equations H_i and H_c that suggests the process (given in the following section) of associating a compressible flow with an incompressible flow.

Since equations (5) are linear, any linear combination of the solutions $W^{(n)}$ and $i\tilde{W}^{(n)}$ are again solutions; that is,

$$\Sigma\left(\alpha_n W^{(n)} + \beta_n i\tilde{W}^{(n)}\right)$$

where α_n and β_n are real constants, is a solution of the system of equations (5). It may be pointed out, furthermore, that the solutions $W^{(n)}$ and $i\tilde{W}^{(n)}$ are of elliptic type in one part of the range of q and of hyperbolic type in the other part of the range of q.

CONVERSION FROM H_i TO P_i AND FROM H_c TO P_c

In the incompressible case the same order of mathematical difficulty exists in going from the solutions of the P_i equations to the solutions of the H_i equations as exists in the reverse process. In the compressible case, however, it is at present necessary to proceed from the solutions of the H_c equations to the solutions of the P_c equations.

Let $\Omega = \phi + i\psi$ represent a flow pattern in the incompressible case. It is known that the reflected velocity vector is

$$\frac{d\Omega}{dz} = qe^{-i\theta}$$

It is convenient to introduce the variable

$$w = i \log \frac{d\Omega}{dz} = \theta + i \log q \qquad (P_i \longrightarrow H_i) \qquad (12)$$

which is a solution of the H_1 equations. Then

$$\frac{d\Omega}{dz} = e^{-iw} \tag{13}$$

If $\Omega(z)$ is regarded as a function of w, equation (13) can be integrated; thus

$$z = \int e^{iw} \frac{d\Omega[z(w)]}{dw}\, dw \qquad (H_1 \rightarrow P_1) \tag{14}$$

Equation (12) may be considered the transformation that converts solutions of the P_1 equations to solutions of the H_1 equations, and equation (14) may be considered as the transformation that converts solutions of the H_1 equations to solutions of the P_1 equations.

As an example, consider the simple case of a source of strength m at the origin in the physical plane. Thus

$$\Omega_1 = \phi + i\psi = \frac{m}{2\pi} \log z \tag{15}$$

Then, from equation (12),

$$w = i \log \frac{m}{2\pi z}$$

or

$$\Omega_1 = \frac{m}{2\pi} iw + \frac{m}{2\pi} \log \frac{m}{2\pi} \tag{16}$$

Thus, Ω_1 as a function of z is given by equation (15) and, as a function of w, by equation (16).

If Ω had been preassigned as a function of w, a solution of the H_1 equations, then by the use of equation

(14) z is obtained as a function of w. Solving for w in terms of z and substituting w back into the preassigned function gives Ω as a function of z. For example, given the relation expressed by equation (16), a solution in the physical plane, from equation (14)

$$z = \frac{m}{2\pi} e^{iw}$$

is obtained. Solving for w in terms of z and substituting in equation (16) gives equation (15), which is identified as a source of strength m. This is the process of going from the solutions of the H_i equations to the solutions of the P_i equations $(H_i \longrightarrow P_i)$.

In the compressible case, the corresponding process of going from the solutions of the H_c equations to the solutions of the P_c equations $(H_c \longrightarrow P_c)$ is followed in this paper. For this purpose a relation analogous to that of equation (14) is necessary. This relationship is given by the pair of known equations:

$$\left.\begin{aligned} dx &= \left(\frac{\cos\theta}{q}\phi_q - \frac{\rho_0 \sin\theta}{\rho q}\psi_q\right)dq + \left(\frac{\cos\theta}{q}\phi_\theta - \frac{\rho_0 \sin\theta}{\rho q}\psi_\theta\right)d\theta \\ dy &= \left(\frac{\sin\theta}{q}\phi_q + \frac{\rho_0 \cos\theta}{\rho q}\psi_q\right)dq + \left(\frac{\sin\theta}{q}\phi_\theta + \frac{\rho_0 \cos\theta}{\rho q}\psi_\theta\right)d\theta \end{aligned}\right\} \quad (17)$$

By simple algebra, equations (17) can be arrived at by use of equations (5). It is useful to bear in mind, solving a particular problem, that equations (17) are exact differentials.

As an example, consider the solution of the H_c equations given by

$$\Omega_c = \phi + i\psi = i\tilde{W} = i\left(\theta + i\int \frac{\rho_0(1-M^2)}{\rho q}\,dq\right) \qquad (18)$$

Here

$$\phi = -\int \frac{\rho_0(1-M^2)}{\rho q}\,dq$$

and

$$\psi = \theta$$

If partial derivatives are taken and substituted in equation (17), the following equations are obtained:

$$\left.\begin{aligned} dx &= -\frac{\rho_0(1-M^2)\cos\theta}{\rho q^2}\,dq - \frac{\rho_0\sin\theta}{\rho q}\,d\theta \\ dy &= -\frac{\rho_0(1-M^2)\sin\theta}{\rho q^2}\,dq + \frac{\rho_0\cos\theta}{\rho q}\,d\theta \end{aligned}\right\} \qquad (19)$$

By use of the relation

$$\frac{d\rho}{dq} = -\frac{\rho M^2}{q}$$

the pair of functions of which equations (19) are the exact differentials are recognized as

$$x = \frac{\rho_0 \cos\theta}{\rho q}$$

$$y = \frac{\rho_0 \sin\theta}{\rho q}$$

If each side of the two equations is squared and summed, the following equation is obtained

$$x^2 + y^2 = r^2 = \rho_o{}^2/\rho^2 q^2 \tag{20}$$

Equation (20) implies that the product of the density and the velocity varies inversely as the distance from the origin.

In order to identify the character of the potential and stream lines, ϕ and ψ are set equal to constants; that is,

$$\phi = -\int \frac{\rho_o(1-M^2)}{\rho q}\, dq = c_1$$

and

$$\psi = \theta = c_2$$

The stream lines

$$\theta = \text{constant}$$

are radial lines, and the potential lines

$$\int \frac{\rho_o(1-M^2)}{\rho q}\, dq = \text{constant}$$

are concentric circles. This flow pattern is therefore that of a source.

It is of interest to observe that by a similar process the solution

$$\Omega_c = W = \theta + i\int \frac{\rho}{\rho_o q}\, dq$$

can be shown to represent a vortex in a compressible fluid in the physical plane.

In the incompressible case, the source in the θ,q-plane is given by iw and, in the compressible case, the source in the θ,q-plane is given by $i\tilde{W}$.

The qualitative similarity between the quantities w and W and the quantities iw and $i\tilde{W}$ can easily be recognized. It is this qualitative similarity between the solutions of the H_i equations and the H_c equations that serves, in a sense, to pick out the "useful" solutions of the H_c equations from the unlimited number of solutions given by the expression

$$\Sigma\left(\alpha_n W^{(n)} + \beta_n i\tilde{W}^{(n)}\right)$$

parametric in α_n and β_n. The following procedure may be used: Given a "useful" flow pattern Ω_i of an incompressible fluid in the physical plane, convert this pattern into a pattern in the H_i plane; that is, convert Ω_i into a function of w and iw. Expand Ω_i in a power series:

$$\Omega_i = \Sigma\left(\alpha_n w^n + \beta_n iw^n\right) \tag{21}$$

Then, the compressible flow given by

$$\Omega_c = \Sigma\left(\alpha_n W^{(n)} + \beta_n i\tilde{W}^{(n)}\right) \tag{22}$$

where the real constants α_n and β_n are the same as in equation (21), is the associated compressible-flow pattern of the incompressible-flow pattern given by equation (21).

If equation (21) represents an incompressible flow past a body B_i, then equation (22) represents a compressible flow past an associated body B_c, which may be distorted from B_i by some factor depending on the Mach number. When the body B_c is obtained, well-known methods (reference 2) can be used to find the incompressible flow past B_c in order that the two flow patterns about B_c can be compared and studied for a given Mach number.

If the compressible flow about a preassigned body is desired, it will be necessary to start with a body B_i that is distorted in the opposite direction.

Finally, it is emphasized that, in order to obtain a suitable computational procedure for the process mentioned in this paper, the functions corresponding to the elementary functions - for example, sine, exponential, and logarithm - must be studied and tabulated. Some information concerning these functions can be found in reference 1.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va.

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